A MULTIGRID METHOD FOR THE SOLUTION OF COMPOSITE MESH PROBLEMS

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***Abstract*— The Composite Finite Element Mesh method is useful for the estimation of the discretization error and, in addition, for the nodal solution improvement with a small increase in the computational cost. The technique uses two meshes with different element size to discretize a given problem and, then, it redefines the resulting linear system. On the other hand, Multigrid methods solve a linear system using systems of several sizes resulting from a hierarchy of meshes. This feature motivates the study of the application of the Multigrid strategy together with the Composite Mesh technique. In this work, it is proposed a Multigrid method to solve problems where the Composite Mesh is applied. The goal of the proposal is to achieve both, the advantages of the Multigrid algorithm efficiency and the solution improvement given by the Composite Mesh technique. The new method is tested with some elliptic problems with analytical solution.**

***Keywords*— Multigrid method, Composite Mesh, Numerical solution improvement.**

I. INTRODUCTION

In this work, a Multigrid (MG) technique able to solve the linear system arising from the application of the Composite Mesh (CM) strategy is presented. The application of the Finite Element Method (FEM) to discretize a partial differential equation leads to a linear system, which could be solved with some MG strategy in a very efficient way. Of course, the element size used to discretize the problem domain directly affects both the computational cost, through the size of the linear system, and the error level. For a given problem, the idea behind the CM strategy is to perform a linear combination between the discrete systems of equations computed with FEM using two meshes of different element size. These meshes must have nodes in common. Typically, one mesh is the homogeneous refinement of the other one. With an appropriate choice of the coefficients of the linear combination, the CM technique could give a better nodal solution than those obtained from each mesh individually without increasing the computational cost significantly (Bergallo *et al.*, 2000). Although the CM method has been tested for several kind of problems (Sarraf, 2011), the best performance of the strategy was found for elliptic problems where the solution has a high degree of regularity (Sonzogni *et al.*, 1996; Bergallo *et al.*, 2000).

The MG method is a well known strategy useful for solving differential equations using a hierarchy of meshes defined over the problem domain (Lallemand *et al.*, 1992; Wesseling, 1992; Koobus *et al.*, 1994; Venkatakrishnan and Mavriplis, 1994; Mavriplis, 1995; Chan *et al.,* 1997; Briggs *et al.*, 2000; Arnold, 2001; Okusanya, 2002; Kim *et al.,* 2004). MG algorithms are applied to a wide range of problems, primarily to solve linear and nonlinear boundary value problems. One of the first applications of the MG techniques was to elliptic problems, which remain today as one of the typical applications of the method.

Since both MG and CM methods use meshes with different degree of refinement grouped in a hierarchy of grids and are suitable for elliptic problems, the motivation to integrate these strategies arises. In this paper we describe the proposed method, which is applied to some elliptic test problems with analytical solution on unstructured meshes, where discretization errors are analyzed.

II. MULTIGRID METHOD

Let Ω be a bounded domain in $\Re^d$ with boundary $\partial\Omega$, d=1, 2, 3 being the space dimension. Consider the following elliptic differential equation with homogeneous boundary conditions

$$\begin{aligned} Lu &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1}$$

Assume that the operator L is self-adjoint, i.e. $(Lu, v)=(u, Lv)$ for any $u, v \in H \subset L^2(\Omega)$ and that it is positive in the sense that $(Lu, u)>0$ for all $u\in H$, $u\neq 0$, where the subspace H contains smooth functions which vanish on $\partial\Omega$. With these properties, to solve the boundary value problem given by Eq. (1) is formally equivalent to minimize the quadratic functional

$$F(u) \equiv \frac{1}{2}(Lu, u) - (f, u), \quad u \in H \tag{2}$$

The problem can be rewritten in compact form as follows

$$u = \arg \min_{v\in H} F(v), \tag{3}$$

which means to find the argument that minimizes F over all the functions in H.

Given a triangulation of Ω with element size h, denoted by Ω_h, let H^h be the finite subspace of H consisting on the functions u_h which are continuous in Ω, polynomial in each element and vanish on the boundary domain. Then, the discrete problem is written as follows